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Growth of polynomials with prescribed zeros – II

K. K. Dewan[1] and Arty Ahuja[2]

[1]Department of Mathematics, Faculty of Natural Sciences, Jamia Millia Islamia (Central University), New Delhi - 110025, India
[2]GGSSS, VV-II, Delhi-92, under Directorate of Education, GNCT of Delhi, India
E-mail: aarty_ahuja@yahoo.com

Abstract

In this paper we consider a class of polynomials $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \le \mu \le n$, having all its zeros on $|z| = k$, $k \le 1$. Using the notation $M(p,t) = \max_{|z|=t} |p(z)|$, we measure the growth of p by estimating $\left\{ \frac{M(p,t)}{M(p,1)} \right\}^s$ from above for any $t \ge 1$, s being an arbitrary positive integer.

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1 Introduction and Statement of Results

For an arbitrary entire function $f(z)$, let $M(f,r) = \max_{|z|=r} |f(z)|$. As a consequence of maximum modulus principle [5, Vol. I, p. 137, Problem III, 269]) it is known that if $p(z)$ is a polynomial of degree n, then

$$M(p,R) \le R^n M(p,1) \quad \text{for} \quad R \ge 1. \tag{1.1}$$

The result is best possible and equality holds for polynomials having zeros at the origin.

Ankeny and Rivlin [1] considered polynomials not vanishing in the interior of the unit circle and obtained refinement of inequality (1.1) by proving that if $p(z) \ne 0$ in $|z| < 1$, then

$$M(p,R) \le \left(\frac{R^n + 1}{2} \right) M(p,1), \quad R \ge 1. \tag{1.2}$$

The result is sharp and equality in (1.2) holds for $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.

While trying to obtain inequality analogous to (1.2) for polynomials not vanishing in $|z| < k$, $k \le 1$, recently the authors [2] proved the following result.

Theorem A. If $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$, then for every positive integer s

$$\{M(p,R)\}^s \le \left(\frac{k^{n-1}(1+k) + (R^{ns} - 1)}{k^{n-1} + k^n} \right) \{M(p,1)\}^s, \quad R \ge 1. \tag{1.3}$$

By involving the coefficients of $p(z)$, Dewan and Ahuja [2] in the same paper obtained the following refinement of Theorem A.

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Theorem B. If $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then for every positive integer s

$$\{M(p,R)\}^s \leq \frac{1}{k^n}\left(\frac{n|c_n|\{k^n(1+k^2)+k^2(R^{ns}-1)\}+|c_{n-1}|\{2k^n+R^{ns}-1\}}{2|c_{n-1}|+n|c_n|(1+k^2)}\right)\{M(p,1)\}^s, \; R\geq 1. \tag{1.4}$$

In this paper, we consider a class of polynomials $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, having all its zeros on $|z| = k$, $k \leq 1$ and generalize Theorem A and Theorem B. More precisely, we prove

Theorem 1. If $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \leq \mu < n$, is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then for every positive integer s

$$\{M(p,R)\}^s \leq \left(\frac{k^{n-2\mu+1} + k^{n-\mu+1} + R^{ns} - 1}{k^{n-2\mu+1} + k^{n-\mu+1}}\right)\{M(p,1)\}^s, \quad R \geq 1. \tag{1.5}$$

Remark 1. If we choose $\mu = 1$ in Theorem 1,then inequality (1.5) reduces to Theorem A.

For $s = 1$ in Theorem 1, we get the following result.

Corollary 1. If $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \leq \mu < n$, is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then

$$M(p,R) \leq \left(\frac{k^{n-2\mu+1} + k^{n-\mu+1} + R^n - 1}{k^{n-2\mu+1} + k^{n-\mu+1}}\right) M(p,1), \quad R \geq 1. \tag{1.6}$$

The following corollary immediately follows from inequality (1.6) by taking $k = 1$.

Corollary 2. If $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = 1$, then

$$M(p,R) \leq \left(\frac{R^n + 1}{2}\right) M(p,1), \quad R \geq 1. \tag{1.7}$$

If we involve the coefficients of $p(z)$ also, then we are able to obtain a bound which is better than the bound of Theorem 1. In fact, we prove

Theorem 2. If $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \leq \mu < n$, is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then for every positive integer s

$$\{M(p,R)\}^s \leq \frac{1}{k^{n-\mu+1}}\left(\frac{\begin{array}{c} n|c_n|\{k^n(1+k^{\mu+1})+k^{2\mu}(R^{ns}-1)\} \\ +\mu|c_{n-\mu}|\{k^{n-\mu+1}(1+k^{\mu-1})+k^{\mu-1}(R^{ns}-1)\}\end{array}}{\mu|c_{n-\mu}|(1+k^{\mu-1})+n|c_n|k^{\mu-1}(1+k^{\mu+1})}\right)\{M(p,1)\}^s, \; R\geq 1. \tag{1.8}$$

To prove that the bound obtained in the above theorem is, in general, better than the bound obtained in Theorem 1, we show that

$$\frac{1}{k^{n-\mu+1}}\frac{\begin{pmatrix} n|c_n|\{k^n(1+k^{\mu+1})+k^{2\mu}(R^{ns}-1)\} \\ +\mu|c_{n-\mu}|\{k^{n-\mu+1}(1+k^{\mu-1})+k^{\mu-1}(R^{ns}-1)\}\end{pmatrix}}{\mu|c_{n-\mu}|(1+k^{\mu-1})+n|c_n|k^{\mu-1}(1+k^{\mu+1})}$$
$$\leq \frac{k^{n-2\mu+1}+k^{n-\mu+1}+R^{ns}-1}{k^{n-2\mu+1}+k^{n-\mu+1}}$$

which is equivalent to

$$\begin{aligned} &n|c_n|(k^{n-2\mu+1}+k^{n-\mu+1})(k^n+k^{n+\mu+1}+k^{2\mu}R^{ns}-k^{2\mu}) \\ &\quad+\mu|c_{n-\mu}|(k^{n-2\mu+1}+k^{n-\mu+1})(k^n+k^{n-\mu+1}+k^{\mu-1}R^{ns}-k^{\mu-1}) \\ &\qquad\leq n|c_n|(k^n+k^{n+\mu+1})(k^{n-2\mu+1}+k^{n-\mu+1}+R^{ns}-1) \\ &\qquad\quad+\mu|c_{n-\mu}|(k^n+k^{n-\mu+1})(k^{n-2\mu+1}+k^{n-\mu+1}+R^{ns}-1)\end{aligned}$$

which implies

$$\begin{aligned} &n|c_n|(k^{2n-2\mu+1}+k^{2n-\mu+2}+k^{n+1}R^{ns}-k^{n+1}+k^{2n-\mu+1}+k^{2n+2}+k^{n+\mu+1}R^{ns}-k^{n+\mu+1}) \\ &+\mu|c_{n-\mu}|(k^{2n-2\mu+1}+k^{2n-3\mu+2}+k^{n-\mu}R^{ns}-k^{n-\mu}+k^{2n-\mu+1}+k^{2n-2\mu+2}+k^nR^{ns}-k^n) \\ &\quad\leq n|c_n|(k^{2n-2\mu+1}+k^{2n-\mu+1}+k^nR^{ns}-k^n+k^{2n-\mu+2}+k^{2n+2}+k^{n+\mu+1}R^{ns}-k^{n+\mu+1}) \\ &\qquad+\mu|c_{n-\mu}|(k^{2n-2\mu+2}+k^{2n-3\mu+2}+k^{n-\mu+1}R^{ns}-k^{n-\mu+1}+k^{2n-2\mu+1}+k^{2n-\mu+1}+k^nR^{ns}-k^n), \\ &\mu|c_{n-\mu}|\{k^{n-\mu}(R^{ns}-1)-k^{n-\mu+1}(R^{ns}-1)\}\leq n|c_n|\{k^n(R^{ns}-1)-k^{n+1}(R^{ns}-1)\}, \\ &\mu|c_{n-\mu}|k^{n-\mu}\leq n|c_n|k^n, \\ &\frac{\mu|c_{n-\mu}|}{n|c_n|}\leq k^\mu,\end{aligned}$$

which is always true (see Lemma 4).

If we choose $s=1$ in Theorem 2,we get the following result.

Corollary 3. If $p(z)=c_nz^n+\sum_{j=\mu}^{n}c_{n-j}z^{n-j}$, $1\leq\mu<n$, is a polynomial of degree n having all its zeros on $|z|=k$, $k\leq 1$

$$M(p,R)\leq\frac{1}{k^{n-\mu+1}}\left(\frac{\begin{matrix} n|c_n|\{k^n(1+k^{\mu+1})+k^{2\mu}(R^n-1)\} \\ +\mu|c_{n-\mu}|\{k^{n-\mu+1}(1+k^{\mu-1})+k^{\mu-1}(R^n-1)\}\end{matrix}}{\mu|c_{n-\mu}|(1+k^{\mu-1})+n|c_n|k^{\mu-1}(1+k^{\mu+1})}\right)M(p,1), R\geq 1. \tag{1.9}$$

Remark 2. (i) If we choose $\mu=1$ in Theorem 2,then inequality (1.8) reduces to Theorem B.
(ii) For $k=1$ in inequality (1.9), we get Corollary 2.

2 Lemmas

We need the following lemmas for the proof of these theorems. The first lemma is due to Dewan and Hans [3].

Lemma 1. If $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \le \mu < n$, is a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$

$$\max_{|z|=1} |p'(z)| \le \frac{n}{k^{n-2\mu+1} + k^{n-\mu+1}} \max_{|z|=1} |p(z)|. \tag{2.1}$$

Lemma 2. If $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \le \mu < n$, is a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$, then

$$\max_{|z|=1} |p'(z)| \le \frac{n}{k^{n-\mu+1}} \frac{n|c_n|k^{2\mu} + \mu|c_{n-\mu}|k^{\mu-1}}{\mu|c_{n-\mu}|(1 + k^{\mu-1}) + n|c_n|k^{\mu-1}(1 + k^{\mu+1})} \max_{|z|=1} |p(z)|. \tag{2.2}$$

The above lemma is due to Dewan and Hans [4].

Lemma 3. Let $p(z) = c_0 + \sum_{\upsilon=\mu}^{n} c_\upsilon z^\upsilon$, $1 \le \mu \le n$, be a polynomial of degree n having no zero in the disk $|z| < k$, $k \ge 1$,

$$\frac{\mu}{n} \left| \frac{c_\mu}{c_0} \right| k^\mu \le 1\,. \tag{2.3}$$

The above lemma was given by Qazi [6, Remark 1].

Lemma 4. Let $p(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$,

$$\frac{\mu}{n} \left| \frac{c_{n-\mu}}{c_n} \right| \le k^\mu\,. \tag{2.4}$$

Proof of Lemma 4. If $p(z)$ has all its zeros on $|z| = k$, $k \le 1$, then $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$, has all its zeros on $|z| = \frac{1}{k}$, $\frac{1}{k} \ge 1$. Now applying Lemma 3 to the polynomial $q(z)$, Lemma 4 follows.

3 Proof of the theorems

Proof of Theorem 1. Let $M(p, 1) = \max_{|z|=1} |p(z)|$. Since $p(z)$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \le 1$, therefore by Lemma 1, we have

$$|p'(z)| \le \frac{n}{k^{n-2\mu+1} + k^{n-\mu+1}} M(p, 1) \quad \text{for} \quad |z| = 1.$$

Now $p'(z)$ is a polynomial of degree $n - 1$, therefore,it follows by (1.1) that for all $r \ge 1$ and $0 \le \theta < 2\pi$

$$|p'(re^{i\theta})| \le \frac{nr^{n-1}}{k^{n-2\mu+1} + k^{n-\mu+1}} M(p, 1). \tag{3.1}$$

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Also for each $\theta, 0 \leq \theta < 2\pi$ and $R \geq 1$, we obtain

$$\begin{aligned}\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s &= \int_1^R \frac{d}{dt}\{p(te^{i\theta})\}^s dt \\ &= \int_1^R s\{p(te^{i\theta})\}^{s-1} p'(te^{i\theta}) e^{i\theta} dt.\end{aligned}$$

This implies

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leq s\int_1^R |p(te^{i\theta})|^{s-1}|p'(te^{i\theta})|dt,$$

which on combining with inequalities (3.1) and (1.1), we get

$$\begin{aligned}|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| &\leq \frac{ns}{k^{n-2\mu+1}+k^{n-\mu+1}}\{M(p,1)\}^s \int_1^R t^{ns-1}dt, \\ &= \left(\frac{R^{ns}-1}{k^{n-2\mu+1}+k^{n-\mu+1}}\right)\{M(p,1)\}^s.\end{aligned}$$

Therefore,

$$\begin{aligned}|p(Re^{i\theta})|^s &\leq |p(e^{i\theta})|^s + \left(\frac{R^{ns}-1}{k^{n-2\mu+1}+k^{n-\mu+1}}\right)\{M(p,1)\}^s, \\ &\leq \{M(p,1)\}^s + \left(\frac{R^{ns}-1}{k^{n-2\mu+1}+k^{n-\mu+1}}\right)\{M(p,1)\}^s \qquad (3.2)\end{aligned}$$

Hence, from (3.2), we conclude that

$$\{M(p,R)\}^s \leq \left(\frac{k^{n-2\mu+1}+k^{n-\mu+1}+R^{ns}-1}{k^{n-2\mu+1}+k^{n-\mu+1}}\right)\{M(p,1)\}^s.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. The proof of Theorem 2 follows on the same lines as that of Theorem 1 by using Lemma 2 instead of Lemma 1. But for the sake of completeness we give a brief outline of the proof. Since $p(z)$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, therefore, by Lemma 2, we have

$$|p'(z)| \leq \frac{n}{k^{n-\mu+1}}\left(\frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{n|c_n|k^{\mu-1}(1+k^{\mu+1})+\mu|c_{n-\mu}|(1+k^{\mu-1})}\right)M(p,1) \quad \text{for} \quad |z|=1.$$

Now $p'(z)$ is a polynomial of degree $n-1$, therefore, it follows by (1.1) that for all $r \geq 1$ and $0 \leq \theta < 2\pi$

$$|p'(re^{i\theta})| \leq \frac{nr^{n-1}}{k^{n-\mu+1}}\left(\frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{n|c_n|k^{\mu-1}(1+k^{\mu+1})+\mu|c_{n-\mu}|(1+k^{\mu-1})}\right)M(p,1). \qquad (3.3)$$

Also for each θ, $0 \leq \theta < 2\pi$ and $R \geq 1$, we have

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leq s\int_1^R |p(te^{i\theta})|^{s-1}|p'(te^{i\theta})|dt,$$

which on combining with inequalities (1.1) and (3.3), we get

$$\begin{aligned}
&|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \\
&\quad \leq \left(\frac{R^{ns}-1}{k^{n-\mu+1}}\right)\left(\frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{n|c_n|k^{\mu-1}(1+k^{\mu+1})+\mu|c_{n-\mu}|(1+k^{\mu-1})}\right)\{M(p,1)\}^s,
\end{aligned}$$

which implies

$$\begin{aligned}
|p(Re^{i\theta})|^s \quad &\leq \quad \{M(p,1)\}^s + \left(\frac{R^{ns}-1}{k^{n-\mu+1}}\right) \\
&\quad \times \left(\frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{n|c_n|k^{\mu-1}(1+k^{\mu+1})+\mu|c_{n-\mu}|(1+k^{\mu-1})}\right)\{M(p,1)\}^s,
\end{aligned}$$

from which the proof of Theorem 2 follows.

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